Moments' criterion asymptotic normality for testing of statistical hypothesis's

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Abstract: - The work is devoted to the synthesis the new moments' criterion asymptotic normality for testing of statistical hypothesis's. It is shown that the decision functions can be presented by the of stochastic polynomials, optimum coefficient of which are by the criterion asymptotically normality. For description of random variables use moments and kumulants. Usage of this description of random variables gives the opportunity to construct decisive rules, which include Non-Gaussian noise parameters.

Key-Words: – Stochastic polynomials, moments, kumulants, Non-Gaussian noise.

1 Introduction

The problem of signals detection on a noise background is one of the most important in statistical radio engineering. Many approaches to the solution of the given problem are known. They are based on testing of simple statistical hypothesis. It is known, they are based on the decision function which is represented by the comparison of the relation of likelihood to any other value. This value is chosen from criterions of quality (Bayes criterion etc.). Such criterion we shall name as *probability*, since in their basis are the probabilities of errors of the first and second kind.

In theory of probabilities and mathematical statistics of random variables it is possible to describe quantitatively by two ways: either with the determination of probability of implementation of this or that event, or with the help of more rough quantitative measure of the numerical characteristics of random variables - such as expectation, variance etc. Criterions, based on usage of the moments of decision function, shall name the moment a*s criterions*.

The construction of algorithms for detection of radio signals on a Non-Gaussian noise background on probability criterion calls some difficulties. Consideration of other methods for the solution of this problem is of interest.

The method of signals detection, when as decision function the stochastic polynomial 2-nd degrees is used developed [1, 2]. According to this criterion (deflection criterion) the optimum decisive rule is set as a polynomial of a final degree, coefficients of which are received from the minimum significance of the functional

$$D_1 = \frac{(T_1 - T_0)^2}{G_0},$$

where T_i - mathematical expectation of decision function to hypothesis H_i,

G_i - variance of decision function to hypothesis H_i, $i = 0, 1$.

The usage of the given criterion calls some displeasure, as the connection with the well known probability criterions is not shown. The development of a new criterion of quality of testing simple statistical hypothesizes — criterion of an asymptotic normality, based on moment and the cumulant description of random variables is offered [3].

2 The criterion asymptotic normality

The new criteria of quality is offered , which differs from deflection criterion and is connected with probabilistic criterion of the sum probabilities of errors. Let's assume, that the decisive rule looks like

$$f(\vec{x}) = g(\vec{x}) - k_0 \underset{H_0}{\overset{H_1}{\gtrless}} 0, \qquad (1)$$

where $\gamma(\vec{x})$ - some function from selective significance's, k_0 - constant.

Let's assume, that the function $\gamma(\vec{x})$ is distributed (or asymptotically is distributed) under the normal law. The decisive rule should be such to minimize probability criterions. As a quality criterion we shall take a criterion of the errors probabilities' sum

$$R = \alpha + \beta . \tag{2}$$

The decisive rule (1) must be selected to minimize function R (2).

Asymptotically [3] in a general view the errors probabilities' criterion (2) will be noted as

$$R(\alpha,\beta) = \frac{1}{\sqrt{2\pi}} \int_{V_0}^{\infty} \exp\left(-\frac{z^2}{2}\right) dz + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{V_1} \exp\left(-\frac{z^2}{2}\right) dz . \tag{3}$$

For the optimum decisive rule (1) constant k_0, which minimizes (2), looks like

$$k_0 = -\frac{T_1 G_0^{0,5} + T_0 G_1^{0,5}}{G_0^{0,5} + G_1^{0,5}} . \tag{4}$$

For this constant the integration limits named V_0 and V_1 are equal to

$$V_0 = Yu^{-0,5}, \quad V_1 = -Yu^{-0,5},$$

where

$$Y_u[\gamma(\vec{x})] = \frac{\left(G_0^{0.5} + G_1^{0.5}\right)^2}{(T_1 - T_0)^2} . \tag{5}$$

It is shown in (3) and (5), that the significance of the errors probabilities' criterion $R(\alpha,\beta)$ depends only on the functional Y_u. The functional $Y_u[\gamma(\vec{x})]$ can be taken as a criterion quality of the decisive rules testing.

<u>*Definition 1.*</u> Let's accept a functional $Y_u[\gamma(\vec{x})]$ as a criterion of quality of the decisive rules testing of kind (1). Let's consider to be the best rule one which for k_0 (4), minimizes on all possible $\gamma(\vec{x})$ functional $Y_u[\gamma(\vec{x})]$. Let's name this criterion as the criterion of asymptotic normality.

3 The use of loglikelihood ratio as stochastic polynomial

Let's selective values $\vec{x} = \{x_1, \ldots, x_n\}$ by hypothesis H_0 and H_1 are statistically independent. Then the density function of independent random variables for $\vec{x}$ looks like

$$P(\vec{x}/H_k) = \prod_{v=1}^{n} P_v(x_v/H_k), k = 0,1 .$$

The loglikelihood ratio in this case looks like

$$\ln\frac{P(\vec{x}/H_1)}{P(\vec{x}/H_0)} = \sum_{v=1}^{n} \ln\frac{P_v(x_v/H_1)}{P_v(x_v/H_0)} .$$

Let's assume, that loglikelihood ratio for any v is a continuous function concerning each x_v. Then in agreement with the 1-st theorem of the Weierstrass, there are such coefficients k_i, that loglikelihood ratio looks like

$$\ln\frac{P(x_v/H_1)}{P(x_v/H_0)} = k_{0v} + \sum_{i=1}^{\infty} k_{iv} x_v^i .$$

Loglikelihood ratio for n independent unequally distributed random variables will be accordingly equal

$$\ln\frac{P(\vec{x}/H_1)}{P(\vec{x}/H_0)} = k_0 + \sum_{v=1}^{n}\sum_{i=1}^{\infty} k_{iv} x_v^i ,$$

where $k_0 = \sum_{v=1}^{n} k_{0v}$.

If we use not a series but polynomial of a final degree s, then the decisive rule looks like

$$k_0 + \sum_{v=1}^{n}\sum_{i=1}^{s} k_{iv} x_v^i \underset{H_0}{\overset{H_1}{\gtrless}} 0 . \tag{6}$$

If selective values are equally distributed, the decisive rule will differ from obtained in (6). In this case the coefficients k_i do not independ on numbers of selective values v. Then the decisive rule looks like

$$l_{ns}(\vec{x}) = k_0 + \sum_{i=1}^{s} k_i \sum_{v=1}^{n} x_v^i \underset{H_0}{\overset{H_1}{\gtrless}} 0 . \tag{7}$$

The right part of a decisive rule (7) according to the central limiting theorem asymptotically at $n \to \infty$ is distributed under the normal law. Then for determination of coefficients k_i can use the criterion asymptotic normality (5).

The system equations for determination of uncertain coefficients k_i looks like

$$\sum_{j=1}^{s} k_j \left[(1+r)F_{ij}(0) + (1+1/r)F_{ij}(1)\right] = m_i - u_i, \quad (8)$$

where $i = \overline{1,s}, \quad r = (G_1/G_0)^{0.5}$,

$$T_0 = n\sum_{i=1}^{s} k_i u_i, \qquad T_1 = n\sum_{i=1}^{s} k_i m_i,$$

$$G_0 = n\sum_{i=1}^{s}\sum_{j=1}^{s} k_i k_j F_{ij}(H_0),$$

$$G_1 = n\sum_{i=1}^{s}\sum_{j=1}^{s} k_i k_j F_{ij}(H_1),$$

$$F_{ij}(H_0) = u_{i+j} - u_i u_j,$$

$$F_{ij}(H_1) = m_{i+j} - m_i m_j,$$

u_i, m_i - initial moments of random variables ξ for hypothesis H_0 and H_1 accordingly. The minimum values Yu_{sn} looks like

$$Yu_{sn\min} = I_{Yusn}^{-1}.$$

Let's take into consideration some properties, which are characteristic for coefficients k_i.

Property 1. For coefficients k_i, which are found from (8), equality looks like

$$I_{Yusn} = n\sum_{i=1}^{n} k_i (m_i - u_i). \quad (9)$$

Definition 2. Value I_{Yusn} we shall take as quantity of the extracted information in sample with volume *n* about the difference between hypothesis H_0 and H_1 with the help of stochastic decisive rule (7). This decisive rule is optimum by the criterion asymptotic normality.

Property 2. For coefficients k_i, which are found from system equations (8), inequality looks like

$$I_{Yusn} = \sum_{i=1}^{\infty} k_i (m_i - u_i) \geq 0.$$

In agreement with (5) and property 1, the criterion asymptotic normality looks like

$$Yu_{sn} = \frac{1}{I_{Yun}},$$

i.e. criterion asymptotic normality is inversely proportional to quantity of the extracted information about difference between of hypotheses.

On the basis of the criterion asymptotic normality the construction of signals detectors on a background Non-Gaussian noise is possible.

4 Conclusion

The new method to construct of the decisive rules, based on loglikelihood ratio in kind of stochastic polynomials is offered in this work. The optimum coefficient of such decisive rules are obtained by criterion asymptotic normality with the help of moment and cumulant description of random variables. The given method allows to receive better characteristics for development of algorithms of signals detection on a Non-Gausisan noise background.

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